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# The Boundary Element Method for the Solution of the Backward Heat Conduction Equation

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In this paper we consider the numerical solution of the one-dimensional, unsteady heat conduction equation in which Dirichlet boundary conditions are specified at two space locations and the temperature distribution at a particular time, say $T_0$, is given. The temperature distribution for all times, $t < T_0$, is now required and this backward heat conduction problem is a well-known improperly posed problem. In order to solve this problem the minimal energy technique has been introduced in order to modify the boundary element method and this results in a stable approximation to the solution and the accuracy of the numerical results are very encouraging. © 1995 Academic Press, Inc.

## 1. INTRODUCTION

The aim of this paper is to investigate an improperly posed problem that arises in one-dimensional, unsteady heat transfer, but before proceeding we recall what is meant by an improperly posed problem in partial differential equations. One may regard a problem as being well posed (or properly posed) if a unique solution exists which depends continuously on the data; otherwise it is an improperly posed problem. Of course, we must state precisely in what class the solution is to lie as well as the measure of continuous dependence. It should be emphasized that in discussing the continuous dependence on the data we must consider the initial and boundary conditions, the prescribed values of the operator, the coefficients of the terms in the governing equation and the geometry of the solution domain.

The systematic study of classes of improperly posed problems for partial differential equations is of rather recent origin, although consideration was already being given to such problems in the middle and latter half of the nineteenth century. Hadamard [6] studied the Cauchy problem for the Laplace equation and he clearly defined what is meant by an improperly posed problem and illustrated by examples and counterexamples the difficulties involved. Hadamard further pointed out that it is impossible to solve an improperly posed problem by the classical theory of partial differential equations and he derived the necessary and sufficient conditions for the global existence of solutions of the Cauchy problem for the Laplace equation. Unfortunately, it is impossible to verify, in general, whether the necessary conditions are or are not satisfied. Up to now, a number of procedures have been advanced for the solution of improperly posed problems. Lavrentiev [10] discussed bounded solutions of the Cauchy problem for the Laplace equation in a special two-dimensional domain such that the Cauchy data is continuous. Whilst Payne [13, 14] obtained solutions of more general second-order elliptic equations. Then Falk and Monk [3] investigated error estimates of a regularization method for approximating the Cauchy problem for the Poisson equation on a rectangle. Han [7] studied an energy-bounded solution of the second-order elliptic equations and proposed a minimal energy method for getting its numerical approximation by the finite element method.

An important class of improperly posed problems arises in the field of inverse heat conduction problems and it has been found that these problems are extremely sensitive to measurement errors; see, for example, Beck *et al.* [1]. The unsteady heat conduction problem is generally described by a parabolic equation and if initial and boundary data are specified then this, in general, leads to a well-posed problem, since it is possible to obtain the temperature distribution at any later time uniquely. However, in many physical situations it is not always possible to specify the initial data or the boundary condition at all points on the boundary of region. For example, in practice it may be possible to determine the temperature distribution at a particular time, say $t = T_0 > 0$, and to specify either the temperature, $u$, or the heat flux, $\partial u/\partial n$, on the boundary of the region. From this data the question arises as to whether the temperature distribution at any earlier time, $t < T_0$, can be obtained. In general, no solution which satisfies this equation and the boundary conditions exists. Further, even if a solution did exist it would not be dependent continuously on the boundary and

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initial data; see Payne [5]. Therefore, this problem is an example of an improperly posed problem and in order to solve this problem classical numerical methods are impossible and a special technique has to be employed; see Hadamard [6].

In order to solve inverse heat conduction problems which are improperly posed a number of procedures have been developed. Miller [12] described a least-squares scheme in order to investigate the bounded solution of the backward parabolic problems. Tikhonov and Arsenin [16] have investigated unsteady heat conduction problems in which there are insufficient boundary conditions. They described the regularization method to reduce the sensitivity of improperly posed problems to measurement errors and the singular-value decomposition technique has also been successfully applied to inverse heat transfer problem by Mandrel [11]. Then Ingham *et al.* [8] introduced a minimal energy scheme to modify the boundary element method (BEM) in order to solve a problem in which there was insufficient boundary data in order to determine the solution of the Laplace equation, and an excellent approximate solution was obtained. Further, Ingham and Yuan [9] investigated a steady state, nonlinear heat conduction equation in which there are some unknown coefficients, but extra information was given at some interior points of the solution domain. They have successfully used the minimal energy technique, along with the BEM, to determine those unknown coefficients. In this paper we extend this minimal energy technique to solve the backward unsteady heat conduction problem.

The time dependent heat conduction problem satisfies the equation

$$\partial u(x, t)/\partial t = \nabla^2 u(x, t) \quad (x, t) \in \Omega \times (0, \infty), \tag{1.1}$$

where $u(x, t)$ is the temperature. If the temperature, or heat flux, is specified on the surface of the domain $\Omega$ and the initial temperature distribution is given, then Eq. (1.1) has to be solved, subject to the conditions

$$u(x, t) = \phi(x, t) \quad (x, t) \in \partial\Omega \times [0, \infty) \tag{1.2}$$

$$u(x, 0) = u_0(x), \quad x \in \Omega, \tag{1.3}$$

where $\phi(x, t)$ and $u_0(x)$ are prescribed functions. It is well known that if $\phi$ and $u_0$ are sufficiently smooth then the problem (1.1)–(1.3) has a unique and stable solution; see, for example, Treves [17]. Such problems may easily be solved numerically by using the finite element method, the finite difference method, or the BEM. However, frequently in practice the temperature distribution is prescribed at a particular time, $t = T_0 > 0$, and it is required to find the temperature distribution, or heat flux history, for any time $t < T_0$. This is called the backward heat conduction problem. This problem is much more difficult to solve than the forward heat conduction problem, because the backward problem is improperly posed. In other words, the difficulty is that the solution of the backward heat conduction problem does not continuously depend on the known boundary and initial data.

For simplicity, in this paper we consider the one-dimensional backward heat conduction problem, namely,

$$\partial u/\partial t = \partial^2 u/\partial x^2 \quad (x, t) \in (0, 1) \times (0, T] \tag{1.4}$$

$$u(0, t) = \phi_0(t) \quad t \in [0, T] \tag{1.5}$$

$$u(1, t) = \phi_1(t) \quad t \in [0, T] \tag{1.6}$$

$$u(x, T_0) = g(x) \quad x \in [0, 1], \tag{1.7}$$

where $\phi_0(t)$, $\phi_1(t)$, and $g(x)$ are prescribed functions and the value of $T_0$ ($\leq T$) is given. In order to obtain the numerical solution of the problem (1.4)–(1.7) a constant BEM has been employed and three mathematical models, namely, the direct method, the least-squares method, and the minimal energy method, have been developed. However, it has been found that only the minimal energy technique gives accurate and stable approximate solutions to the problem.

## 2. THE BOUNDARY ELEMENT METHOD

The crucial step in the application of the BEM is the transformation from the governing differential equation to an integral representation and this is achieved by employing a fundamental solution of the differential equations. It is well known that the one-dimensional time-dependent fundamental solution of Eq. (1.4) is of the form, see, for example, Brebbia [2],

$$F(x, t; \xi, \tau) = \frac{1}{\{4\pi(t - \tau)\}^{1/2}} \exp\left[-\frac{(x - \xi)^2}{4(t - \tau)}\right] H(t - \tau), \tag{2.1}$$

where $H(\tau)$ is the Heaviside function which is included to emphasize the fact that the fundamental solution is identically zero for $\tau > t$. Using this fundamental solution then the differential equation (1.4) may be transformed into the boundary integral equation

$$\begin{aligned} \eta(x)u(x, t) = &\int_0^t \phi'(\xi, \tau)F(x, t; \xi, \tau)\Big|_0^1 d\tau \\ &- \int_0^t \phi(\xi, \tau)F'(x, t; \xi, \tau)\Big|_0^1 d\tau \\ &+ \int_0^1 u_0(y)F(x, t; y, 0)\, dy, \end{aligned} \tag{2.2}$$

where $t \in [0, T]$, $\phi = \phi_0$ when $\xi = 0$ and $\phi = \phi_1$ when $\xi = 1$, and

$$\eta(x) = \begin{cases} 1 & \text{when } 0 < x < 1 \\ \frac{1}{2} & \text{when } x = 0, 1 \\ 0 & \text{when } x \notin [0, 1] \times [0, T] \end{cases}$$

In practice analytical solutions of the integral equation (2.2) is impossible and thus some form of numerical approximation is necessary. If the time dimension is subdivided into $N$ time steps, the space interval [0, 1] into $N_0$ elements, and if we take $x$ on the boundary, at $x = 0$ or $x = 1$, then the integral equation (2.2) becomes

$$\frac{1}{2}u(x, \tilde{t}_i) = \sum_{j=1}^{i} \int_{t_{j-1}}^{t_j} \phi'(\xi, \tau)F(x, \tilde{t}_i; \xi, \tau)\Big|_0^1 d\tau - \sum_{j=1}^{i} \int_{t_{j-1}}^{t_j} \phi(\xi, \tau)F'(x, \tilde{t}_i; \xi, \tau)\Big|_0^1 d\tau + \sum_{j=1}^{N_0} \int_{y_{j-1}}^{y_j} u_0(y)F(x, \tilde{t}_i; y, 0)\, dy, \quad i = 1, ..., N, \tag{2.3}$$

where $\tilde{t}_i$ is the midpoint of the segment $[t_{i-1}, t_i]$. Furthermore, if we write

$$G_{ij} = \int_{t_{j-1}}^{t_j} F(x_i, \tilde{t}_i; \xi, \tau)\, d\tau, \quad i = 1, ..., 2N, \tag{2.4}$$

$$E_{ij} = \int_{t_{j-1}}^{t_j} F'(x_i, \tilde{t}_i; \xi, \tau)\, d\tau + \eta_i\delta_{ij}, \quad i = 1, ..., 2N, \tag{2.5}$$

$$F_{ij} = \int_{y_{j-1}}^{y_j} F(x_i, \tilde{t}_i; y, 0)\, dy, \quad i = 1, ..., 2N, \tag{2.6}$$

where $x_i = 0$ for $i \le N$, $x_i = 1$ for $i > N$, and $\xi = 0$ for $j \le N$, $\xi = 1$ for $j > N$. Then from the integral equation (2.3) we obtain a linear system of equations

$$\sum_{j=1}^{2N} G_{ij}\phi_j - \sum_{j=1}^{2N} E_{ij}\phi'_j + \sum_{j=1}^{N_0} F_{ij}u_{0j} = 0, \quad i = 1, ..., 2N, \tag{2.7}$$

where $\phi_j$ and $\phi'_j$ take the values of $\phi$ and $\phi'$ at the midpoint of the segment $[t_{j-1}, t_j]$ and $u_{0j}$ takes the value of $u_0$ at the midpoint of the segment $[y_{j-1}, y_j]$. We should note that $G_{ij} = E_{ij} = 0$ when $t_j > t_i$. The system of Eqs. (2.7) contains $2N$ equations and $4N + N_0$ variables. If either $\phi$ or $\phi'$ is specified at each mesh point on the boundaries $x = 0$ and $x = 1$ and the initial condition $u_0$ is given on $t = 0$ then the system of Eqs. (2.7) can be solved and the temperature distribution can be determined at any point within the solution interval at any particular value of the time $t \in [0, T]$.

All the integrals that occur in expressions (2.4)–(2.6) are calculated using Gaussian quadrature but it should be noted that when $t_j = t_i$ then the integrals (2.4) and (2.5) are singular. However, these integrals may be calculated analytically as follows:

Let $\tilde{t}_i$ be the midpoint of the segment $[t_{i-1}, t_i]$; then we have

$$F(x, \tilde{t}_i; \xi, \tau) = \frac{1}{\{4\pi(\tilde{t}_i - \tau)\}^{1/2}} \exp\left[-\frac{(x - \xi)^2}{4(\tilde{t}_i - \tau)}\right] \tag{2.8}$$

$$F'(x, \tilde{t}_i; \xi, \tau) = \frac{(-1)^x(\xi - x)}{4\pi^{1/2}(\tilde{t}_i - \tau)^{3/2}} \exp\left[-\frac{(x - \xi)^2}{4(\tilde{t}_i - \tau)}\right], \tag{2.9}$$

where $x = 0$ or 1, and $\xi = 0$ or 1. Therefore $F'(x, \tilde{t}_i; x, \tau) \equiv 0$ and $F(x, \tilde{t}_i; x, \tau) = 1/[2\pi^{1/2}(\tilde{t}_i - \tau)^{1/2}]$; then we obtain

$$G_{ii} = \int_{t_{i-1}}^{t_i} F(x, \tilde{t}_i; x, \tau)\, d\tau = ((\tilde{t}_i - t_{i-1})/\pi)^{1/2} \tag{2.10}$$

$$E_{ii} = \int_{t_{i-1}}^{t_i} F'(x, \tilde{t}_i; x, \tau)\, d\tau + \frac{1}{2} = \frac{1}{2}. \tag{2.11}$$

Further, let $\sigma = 1/(\tilde{t}_i - \tau)^{1/2}$ and thus $d\tau = -2d\sigma/\sigma^3$; then we obtain

$$\begin{aligned} E_{iN+i} &= -\int_{t_{i-1}}^{t_i} \frac{1}{4\pi^{1/2}(\tilde{t}_i - \tau)^{3/2}} \exp\left[-\frac{1}{4(\tilde{t}_i - \tau)}\right] d\tau \\ &= \frac{1}{2\pi^{1/2}} \int_{\alpha}^{\infty} \exp[-\sigma^2/4]\, d\sigma \\ &= \frac{1}{2} - \int_0^{\alpha} \exp[-\sigma^2/4]\, d\sigma, \end{aligned} \tag{2.12}$$

where $\alpha = 1/(\tilde{t}_i - t_{i-1})^{1/2}$. On the other hand,

$$\begin{aligned} G_{iN+i} &= -\int_{t_{i-1}}^{t_i} \frac{1}{2\pi^{1/2}(\tilde{t}_i - \tau)^{1/2}} \exp\left[-\frac{1}{4(\tilde{t}_i - \tau)}\right] d\tau \\ &= \frac{1}{\pi^{1/2}} \int_{\alpha}^{\infty} \sigma^{-2} \exp[-\sigma^2/4]\, d\sigma \\ &= C - \frac{1}{2\pi^{1/2}} \int_{\alpha}^{\infty} \exp[-\sigma^2/4]\, d\sigma \\ &= C - \frac{1}{2} + \int_0^{\alpha} \exp[-\sigma^2/4]\, d\sigma, \end{aligned} \tag{2.13}$$

where $C = ((\tilde{t}_i - t_{i-1})/\pi)^{1/2} \exp(-\frac{1}{4}(\tilde{t}_i - t_{i-1}))$. Similarly, we have

$$E_{N+ii} = \frac{1}{2} - \int_0^{\alpha} \exp[-\sigma^2/4]\, d\sigma \tag{2.14}$$

and

$$G_{N+ii} = C - \frac{1}{2} + \int_0^{\alpha} \exp[-\sigma^2/4]\, d\sigma. \tag{2.15}$$

It should be noted that in the calculation of the integrals (2.4)–(2.6) the Gaussian quadrative can also be inaccurate when $t_i$ is close to $t_j$. However, a sufficiently large number of elements have been taken, so that this error is not significant in all the examples investigated in this paper.

## 3. MATHEMATICAL MODELS

In order to use the BEM to solve the backward heat conduction problem (1.4)–(1.7) numerically the problem is first discretised and the linear system of Eqs. (2.7), which contains $2N$ equations and $2N + N_0$ unknown variables, is obtained. Thus in order to solve this system of equations we must add further $N_0$ equations and these are obtained using Eq. (1.7) for the known values of $u(x, t)$. Therefore, on using Eq. (2.2) we obtain

$$\begin{aligned} u(x_i, T_0) = & \int_0^{T_0} \phi'(\xi, \tau) F(x_i, T_0; \xi, \tau)\Big|_0^1 \, d\tau \\ & - \int_0^{T_0} \phi(\xi, \tau) F'(x_i, T_0; \xi, \tau)\Big|_0^1 \, d\tau \qquad (3.1) \\ & + \int_0^1 u_0(y) F(x_i, T_0; y, 0) \, dy, \end{aligned}$$

where $x_i$ is the midpoint of the $i$th segment on $t = T_0$. If we write

$$GI_{ij} = \int_{t_{j-1}}^{t_j} F(x_i, T_0; \xi, \tau) \, d\tau, \quad i = 1, \ldots, N_T, \qquad (3.2)$$

$$EI_{ij} = \int_{t_{j-1}}^{t_j} F'(x_i, T_0; \xi, \tau) \, d\tau, \quad i = 1, \ldots, N_T, \qquad (3.3)$$

$$FI_{ij} = \int_{y_{j-1}}^{y_j} F(x_i, T_0; y, 0) \, dy, \quad i = 1, \ldots, N_T, \qquad (3.4)$$

then, from Eq. (3.1), we obtain the $N_T$ linear equations

$$\sum_{j=1}^{2N} GI_{ij}\phi_j - \sum_{j=1}^{2N} EI_{ij}\phi_j' + \sum_{j=1}^{N_0} FI_{ij}u_{0j} = g_i, \quad i = 1, \ldots, N_T, \qquad (3.5)$$

where the quantities $g_i$ take the value of $g(x)$ at the midpoint of the segment $[y_{j-1}, y_j]$.

### 3.1. *Direct Method*

Taking $N_T = N_0$, then the system of Eqs. (3.5) provides $N_0$ equations which when combined with the system of Eq. (2.7) gives a new system of $2N + N_0$ linear equations with $2N + N_0$ unknown variables. We may now solve directly the system of Eqs. (2.7) and (3.5) to obtain $\phi_j'$ ($j = 1, 2, \ldots, 2N$) and $u_{0j}$ ($j = 1, 2, \ldots, N_0$). By inserting these values into Eq. (2.2) we are able to obtain the function $u(x, t)$ everywhere in $[0, 1] \times [0, T]$. However, it was found that the system of Eqs. (2.7) and (3.5) are ill-conditioned and hence no solution of the system of equations is possible. This conclusion is as expected since the problem (1.4)–(1.7) is improperly posed.

### 3.2. *Least-Squares Method*

Because the system of Eqs. (2.7) and (3.5) are ill-conditioned a least-squares technique was investigated. Further, in order to obtain more information from the known data we may take $N_T \geq n_0$, then the system of Eqs. (2.7) and (3.5) become overdetermined. Unfortunately, it is impossible to obtain an accurate stable solution by solving the system of Eqs. (2.7) and (3.5) using this technique and this is, again, probably because we have not added any further restrictions to the solution; see Hadamard [6] and Ingham *et al.* [8]. Examples illustrating the results obtained using this method are given in Section 4.

### 3.3. *Minimization of Energy Method*

Since the direct and the least-squares methods do not give accurate and stable solutions for the backward heat conduction problem (1.4)–(1.7) then another method, namely the minimization of energy method, is introduced. In this method we do not consider the problem (1.4)–(1.7) but rather the related problem

$$\begin{aligned} &\partial u/\partial t = \partial^2 u/\partial x^2, && (x, t) \in (0, 1) \times (0, T] \\ &u(0, t) = \phi_0(t), && t \in [0, T] \\ &u(1, t) = \phi_1(t), && t \in [0, T] \\ &|u(x, T_0) - g(x)| \leq \varepsilon, && x \in [0, 1], \end{aligned} \qquad (3.6)$$

where $\varepsilon$ is a preassigned small quantity. Clearly the problem (3.6) has many solutions and the solution of the problem (1.4)–(1.7) is included in the solution set of problem (3.6). The difficulty now arises as how to determine the one solution from the set of all possible solutions to the problem (3.6) in a way which is stable.

Let $H^1[0, 1]$ and $H^{-1}[0, 1]$ denote the usual Sobolev spaces, and defining a Hilbert space $\Phi$ such that for any $w \in \Phi$ we have

$$w \in L^2(0, T; H^1[0, 1]), \quad \partial w/\partial t \in L^2(0, T; H^{-1}[0, 1]) \quad (3.7)$$

whose norm is

$$\|w(x, t)\| = \left( \int_0^T \|w(\cdot, t)\|_1^2 + \|\partial w(\cdot, t)/\partial t\|_{-1}^2 \, dt \right)^{1/2}, \quad (3.8)$$

where $\|\cdot\|_1$ and $\|\cdot\|_{-1}$ are the norm of the spaces $H^1[0, 1]$ and $H^{-1}[0, 1]$, respectively.

We now consider the initial boundary value problem

$$\begin{aligned} \partial u/\partial t &= \partial^2 u/\partial x^2 \\ u(0, t) &= \phi_0(t) \\ u(1, t) &= \phi_1(t) \\ u(x, 0) &= \zeta(x) \end{aligned} \qquad (3.9)$$

where $\phi_0(t)$, $\phi_1(t)$, and $\zeta(x)$ are given functions. It is well known, see, for example, Treves [17], that if $\phi_0$ and $\phi_1 \in L^2[0, T]$, for any $\zeta(x) \in L^2[0, 1]$ then there is a unique weak solution such that $u(x, t) \in \Phi$ and $u(x, t)$ converges to $\zeta(x)$ as $t \to +0$. Hence we can define an operator

$$\mathbb{A}: L^2[0, 1] \to \Phi$$
$$\mathbb{A}\zeta = u(x, t), \tag{3.10}$$

where $u(x, t)$ is the weak solution of the problem (3.9). If there is a $\zeta \in L^2[0, 1]$ such that $\mathbb{A}\zeta|_{t=T_0} = g(x)$ or $\|\mathbb{A}\zeta|_{t=T_0} - g(x)\|$ is sufficiently small and $\zeta$ continuously depends on the known function $g(x)$, then we may take $u_0(x) = \zeta$ as the approximate solution on $t = 0$ to the problem (1.4)–(1.7).

To determine the function $\zeta$ we now introduce a minimization scheme. This is achieved by multiplying Eq. (1.4) by $u(x, t)$, and on integrating by parts we obtain

$$\frac{1}{2}\int_0^1 u^2(x, t)\,dx + \int_0^t \int_0^1 |\partial u(x, \tau)/\partial x|^2\,dx\,d\tau - \int_0^t u(1, t)\frac{\partial u(1, \tau)}{\partial x}\,d\tau - \int_0^t u(0, t)\frac{\partial u(0, \tau)}{\partial x}\,d\tau = \frac{1}{2}\int_0^1 u^2(x, 0)\,dx, \tag{3.11}$$

where the right-hand side denotes the initial inner energy of the system and the left-hand side describes the inner energy at time $t$, the kinetic energy and absorbed (or released) thermal energy through both ends.

There is no rigorous mathematical argument as to what is the best choice of the energy functional. Therefore we have considered several different choices of $J(u)$ and programmed them to see which one can give the most satisfactory approximate solution to the problem; for example, we have considered

$$\text{(i)}\quad J(u) = \int_0^1 u^2(x, 0)\,dx \tag{3.12}$$

$$\text{(ii)}\quad J_1(u) = J(u) + \int_0^t u(1, t)\frac{\partial u(1, \tau)}{\partial x}\,d\tau + \int_0^t u(0, t)\frac{\partial u(0, \tau)}{\partial x}\,d\tau \tag{3.13}$$

$$\text{(iii)}\quad J_2(u) = \int_0^1 (u^2(x, 0) + |\partial u(x, 0)/\partial x|^2)\,dx. \tag{3.14}$$

Clearly, the energy functional $J(u)$ is the simplest functional to take and we found that we could obtain an accurate and stable solution using this energy functional. The energy functional (3.13) appears to be more appropriate but the results obtained when using this functional were no more accurate than those obtained when using the energy functional (3.12). The use of $J_2(u)$ appears to be a better choice in that it should be able to control high-frequency variations in the amplitude of the function $u_0(x)$. However, when implementing this technique the finite difference method has to be employed in order to evaluate the function $\partial u(x, 0)/\partial x$, which appears in the energy functional, and therefore the method may not be as accurate as when using $J(u)$ as the energy functional. Therefore in all the results presented in this paper $J(u)$ has been taken as the energy functional and we consider the constrained minimal problem

$$J(u) = \inf_{v \in S} J(v), \tag{3.15}$$

where $S = \{v;\ v = \mathbb{A}\zeta \in \Phi,\ |v - g| \le \varepsilon\}$ is a closed convex set in $\Phi$. Let $K = \{\zeta;\ \zeta \in L^2,\ |\mathbb{A}\zeta - g| \le \varepsilon\}$ then expression (3.15) may be rewritten in the form

$$J(\mathbb{A}\zeta) = \inf_{\chi \in K} J(\mathbb{A}\chi) \tag{3.16}$$

which on discretisation becomes

$$J(u) = \frac{1}{2}\sum_{i=1}^{N_0} \int_{x_{i-1}}^{x_i} \zeta^2(x)\,dx. \tag{3.17}$$

The constraint condition in Eq. (3.16) for $\chi \in K$ may be written

$$\left|\sum_{j=1}^{2N} (GI_{il}G_{lm}^{-1}E_{mj} - EI_{ij})\phi_j + \sum_{j=1}^{N_0} (FI_{ij} - GI_{il}G_{lm}^{-1}F_{mj})\zeta_j - g_i\right| \le \varepsilon, \quad i = 1, \ldots, N_T, \tag{3.18}$$

and using the notation $\sum_{j=1}^{2N} (GI_{il}G_{lm}^{-1}E_{mj} - EI_{ij})\phi_j - g_i = \bar{g}_i$ and $W_{ij} = FI_{ij} - GI_{il}G_{lm}^{-1}F_{mj}$ then we have

$$\left|\sum_{j=1}^{N_0} W_{ij}\zeta_j - \bar{g}_j\right| \le \varepsilon. \tag{3.19}$$

The problem now reduces to finding $\zeta = (\zeta_1, \zeta_2, \ldots, \zeta_{N_0})$ which satisfies the constrained minimal problem (3.17) and (3.19) and this is solved using the NAG routine E04UCF. This routine is designed to minimize an arbitrary smooth function, subject to certain constraints which may include simple bounds on the variables, linear constraints, and smooth nonlinear constraints and the method is a sequential quadratic programming method; see, for example, Fletcher [4] and Gill *et al.* [5].

## 4. NUMERICAL RESULTS

In this paper we have described three mathematical models in order to solve the backward heat conduction problem (1.4)–(1.7) and we now give some examples in order to illustrate the accuracy of the methods. Further, where possible the results are compared with the known analytical solution.

In order to illustrate the rate of convergence of the numerical solution using the minimal energy technique we have investigated the effect of the number of discretisations. In particular,

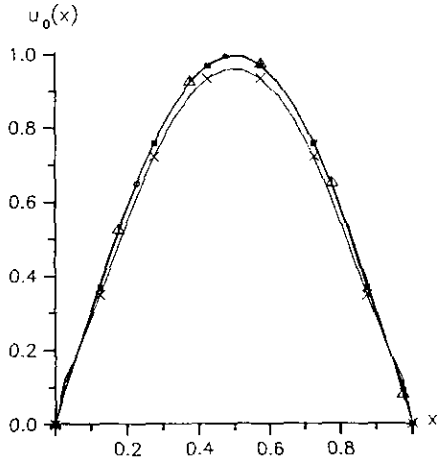


**FIG. 1.** The numerical solutions and the analytical solution $u_0(x)$ for Example 1, where △△△ denote the analytical solution, ××× for $N = 10$, ■■■ for $N = 20$, and ○○○ for $N = 40$.

**TABLE I**

The Numerical Solution at $t = 0$ for Various Values of $x$ for Example 1

| Mesh points ($x$) | Least-squares method | Minimal energy method | Analytical solution |
|---|---|---|---|
| 0.025 | −384937 | 0.09137 | 0.07846 |
| 0.075 | 405915 | 0.22359 | 0.23345 |
| 0.125 | −2490746 | 0.37631 | 0.38268 |
| 0.175 | 2657393 | 0.52090 | 0.52250 |
| 0.225 | −7787272 | 0.64867 | 0.64945 |
| 0.275 | 8298883 | 0.75915 | 0.76041 |
| 0.325 | −15520808 | 0.85110 | 0.85264 |
| 0.375 | 16266417 | 0.92226 | 0.92388 |
| 0.425 | −21738924 | 0.97074 | 0.97237 |
| 0.475 | 22118594 | 0.99530 | 0.99692 |

the effect of the value $N$ is illustrated in Example 1 whilst the effect of $N_0$ and $N_T$ is discussed in Example 2. Further, we have also investigated the effect of the choice of the value of $T_0$ and this is discussed in Example 1. The final example gives an investigation of the stability of an approximate solution using the minimal energy technique.

In all the examples presented the small control quantity $\varepsilon$ has been chosen so as to ensure that the accuracy of the results are not significantly improved when a smaller value is adopted.

EXAMPLE 1. We first take a simple function, $u(x, t) = \sin(\pi x) \exp(-\pi^2 t)$, as the test function and impose the boundary conditions

$$\phi_0(t) = \phi_1(t) = 0 \tag{4.1}$$

and initially at $t = T_0 = \frac{1}{4}$ we specify

$$g(x, \tfrac{1}{4}) = \sin \pi x \exp(-\pi^2/4). \tag{4.2}$$

Solutions of the problem (1.4)–(1.7) with the boundary conditions (4.1) and the final value (4.2) have been obtained with $N_0 = 20$, $N_T = 40$, and $N = 10$, 20, and 40, respectively.

Figure 1 shows the analytical and numerical solution using the minimal energy technique at $t = 0$, where $\varepsilon = 10^{-3}$, $N_0 = 20$, $N_T = 40$, and $N = 10$, 20, and 40, respectively. It is observed that the agreement of the numerical solution as obtained by using the minimal energy technique with the analytical solution is excellent with the maximum relative error being approximately 2%, 1%, and 0.3% for $N = 10$, 20, and 40, respectively. This result illustrates that the minimal energy approximate solution converges as the number of discretisations $N$ increases.

Table I shows the numerical solution as obtained using the least-squares method, the minimal energy method, and the analytical solution at time $t = 0$ for various values of $x$. The numerical solutions have been obtained with $N = 40$, $N_0 = 20$, and $N_T = 40$. It is clear that the least-squares method does not give an accurate approximate solution. Further, the direct method has also been used to solve this problem, but the system of Eq. (2.7) and (3.5) are ill-conditioned. Hence no solution has been obtained by solving the system of Eqs. (2.7) and (3.5) directly. Table II shows the energy of the system at $t = 0$ with $N = 10$, 20, and 40, respectively, and it is observed that the energy of the system when using the minimal technique tends to the analytical solution as $N$ increases, whilst the results obtained using the least-squares method deteriorates as the value of $N$ increases.

We have also investigated the effect of the value of $T_0$. Table III shows the numerical solution using the minimal energy technique with $N_0 = 20$, $N_T = 40$, and in order to compare the accuracy of the numerical results with the same size of segment for various value of $T_0$ then different values of $N$ are used, namely: (i) $N = 40$ for $T_0 = 0.25$; (ii) $N = 80$ for $T_0 = 0.5$; (iii) $N = 160$ for $T_0 = 1.0$, respectively. It is found that there is no significant difference between the accuracy obtained using these three sets of parameters. This result confirms that the

**TABLE II**

The Solution for the Energy Function $J(u)$ at $t = 0$ as a Function of $N$ for Example 1

| $N$ | Least-squares method | Minimal energy method | Analytical value |
|---|---|---|---|
| 10 | $7.148 \times 10^{10}$ | 0.22709 | 0.25 |
| 20 | $6.354 \times 10^{11}$ | 0.24660 | 0.25 |
| 40 | $8.052 \times 10^{13}$ | 0.24883 | 0.25 |

value of $T_0$ is not a very important parameter when using the minimal energy method. We should note that the test function rapidly tends to zero as $T_0$ increases and that obviously zero is a solution of the minimal problem (3.17) if there is no constraint condition. Therefore, in order to avoid this trivial solution and to obtain an accurate approximate solution the control parameter, $\varepsilon$ must be chosen to be very small. In this example it has been found necessary to take $\varepsilon = 10^{-3}$, $10^{-4}$, and $10^{-6}$ for $T_0 = 0.25$, 0.5, and 1.0, respectively, and the use of smaller values of $\varepsilon$, say $10^{-4}$, $10^{-5}$, and $10^{-7}$, respectively, do not have a significant effect on the results as presented in Table III; hence we can conclude that the choice of values of $\varepsilon$ is satisfactory.

EXAMPLE 2. We now take another simple function $u(x, t) = 2t + x^2$ as the test function. This has been taken so as to test the numerical scheme when the boundary condition on $\phi$ is no longer identically zero. Thus we specify

$$\phi_0(t) = 2t, \quad \phi_1(t) = 1 + 2t, \tag{4.3}$$

and

$$u(x, 1) = g(x) = 2 + x^2. \tag{4.4}$$

The solutions of problem (1.4)–(1.7), subject to the conditions (4.3) and (4.4), have been obtained with $N = 40$: (i) $N_0 = 10$ and $N_T = 15$; (ii) $N_0 = 20$ and $N_T = 40$; (iii) $N_0 = 30$ and $N_T = 50$.

Figure 2 shows the numerical solutions at $t = 0$ obtained by using the minimal energy method, with the parameters as given in (i), (ii), and (iii) above, and the analytical solution. The results indicate that the accuracy of the numerical solution is very good and that the numerical solution converges as the number of discretisations $N_0$ and $N_T$ increase.

**TABLE III**

The Numerical Solution at Time $t = 0$ for Various Values of $x$ for Example 1

| $x$ | Numerical solutions $T_0 = 0.25$ | $T_0 = 0.5$ | $T_0 = 1.0$ | Analytical solution |
|---|---|---|---|---|
| 0.025 | 0.09137 | 0.09136 | 0.09135 | 0.07846 |
| 0.075 | 0.22359 | 0.22355 | 0.22354 | 0.23345 |
| 0.125 | 0.37631 | 0.37635 | 0.37632 | 0.38268 |
| 0.175 | 0.52090 | 0.52091 | 0.52091 | 0.52250 |
| 0.225 | 0.64867 | 0.64862 | 0.64862 | 0.64945 |
| 0.275 | 0.75915 | 0.75915 | 0.75912 | 0.76041 |
| 0.325 | 0.85110 | 0.85114 | 0.85110 | 0.85264 |
| 0.375 | 0.92226 | 0.92227 | 0.92223 | 0.92388 |
| 0.425 | 0.97074 | 0.97073 | 0.97068 | 0.97237 |
| 0.475 | 0.99528 | 0.99526 | 0.99522 | 0.99692 |

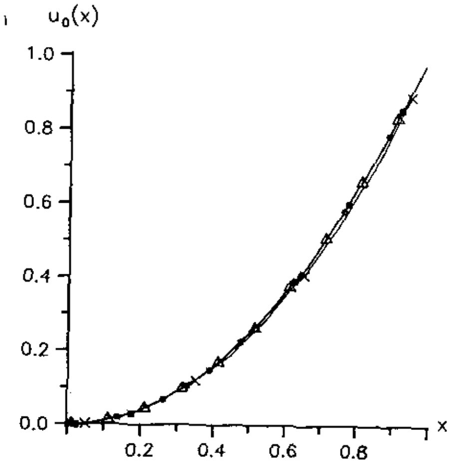


**FIG. 2.** The numerical solutions using the minimal energy method and the analytical solution at $t = 0$ for Example 2, where △△△ is the analytical solution, ××× for $N_0 = 10$, ■■■ for $N_0 = 20$, and ○○○ for $N_0 = 30$.

These two examples have illustrated that numerical solutions of the problem (3.6) using the minimal energy technique are convergent as the number of discretisations increase. However, as we mentioned in the Introduction, the backward heat conduction problem is an improperly posed problem. Therefore, it is necessary to consider the stability of the numerical solution by the minimal energy technique. We again take the test function to be that as given in Example 2 but add a small perturbation $\tilde{g}(x)$ onto the known solution at a given value of $T_0$, i.e., to the function $g(x)$, to see how large an error will be generated by this small perturbation in the initial data.

EXAMPLE 3. We take the boundary conditions as

$$\phi_0(t) = 2t, \quad \phi_1(t) = 1 + 2t \tag{4.5}$$

and the value at $t = 1$ as

$$u(x, 1) = g(x) = 2 + x^2 + \tilde{g}(x), \tag{4.6}$$

where

$$\tilde{g}(x) = \sin(k\pi x) \exp(-k^2\pi^2) \tag{4.7}$$

and $k$ is a constant which we will prescribe a value. It is clear that when $k$ is sufficiently large then the absolute maximum value of the perturbation term $\tilde{g}(x)$ is very small. For example, if we take $k = 3$, then $\max|\tilde{g}(x)| < 10^{-20}$. On the other hand,

$$u(x, t) = 2t + x^2 + \sin(k\pi x) \exp(-k^2\pi^2 t) \tag{4.8}$$

is one of the solutions of the problem (3.6), and at $t = 0$ the value of the term which has arisen from the perturbation is always of the order of unity in the solution (4.8); i.e., the

**TABLE IV**

The Numerical Solutions at $t = 0$ for Various Values of $k$ Using the Minimal Energy Method and the Solution (4.8) for Example 3

| | Numerical solutions | | | Solution (4.8) | | |
|---|---|---|---|---|---|---|
| $x$ | $k = 0$ | $k = 1$ | $k = 3$ | $k = 0$ | $k = 1$ | $k = 3$ |
| 0.025 | 0.00062 | 0.08128 | 0.00062 | 0.00062 | 0.07908 | 0.23407 |
| 0.125 | 0.01548 | 0.39180 | 0.01548 | 0.01562 | 0.39831 | 0.93950 |
| 0.225 | 0.05051 | 0.69918 | 0.05051 | 0.05062 | 0.70007 | 0.90327 |
| 0.325 | 0.10528 | 0.95638 | 0.10528 | 0.10562 | 0.95827 | 0.18408 |
| 0.425 | 0.18041 | 1.05115 | 0.18041 | 0.18062 | 1.15299 | −0.57978 |
| 0.525 | 0.27493 | 1.27023 | 0.27493 | 0.27562 | 1.27254 | −0.69675 |
| 0.625 | 0.39059 | 1.31292 | 0.39059 | 0.39062 | 1.31450 | 0.00794 |
| 0.725 | 0.52551 | 1.28476 | 0.52551 | 0.52562 | 1.28603 | 1.04812 |
| 0.825 | 0.68052 | 1.20151 | 0.68052 | 0.68062 | 1.20312 | 1.67674 |
| 0.925 | 0.85519 | 1.07882 | 0.85519 | 0.85562 | 1.08907 | 1.50507 |

perturbed solution is as large as the unperturbed solution at $t = 0$. Therefore, the solution set of the problem (3.6) includes some solutions which are not continuously dependent on the initial data and this backward heat conduction problem is improperly posed.

Table IV illustrate the numerical solution at $t = 0$ for various values of $x$ and for $k = 0$, 1, and 3, respectively. It is found that the relative error between the two sets of solutions for $k = 0$ and 3 is less than $10^{-7}$. This result, when compared with the solution (3.7) for $k = 3$, gives a relatively large percentage error. However, the results obtained using the minimal energy technique are all within the accuracy of the numerical scheme and clearly the minimal energy method gives a stable approximate solution, since a small perturbation in the input data on $t = T_0$ results in a very small change in the numerical solution.

## 5. CONCLUSIONS

The boundary element method has been modified by a minimal energy technique in order to investigate the backward heat conduction problem which is an improperly posed problem. It has been found that the minimal energy technique always gives an accurate, convergent, and stable solution with an increasing accuracy as the number of discretisations increase. However, the direct and the least-squares methods did not produce accurate solutions to the problem and this is probably because no restrictions on the solution have been enforced and Hadamard [6] affirmed that one has to add some restrictions in order to obtain a solution to improperly posed problems.

It is important to observe when the minimization problem is employed that a good initial guess is not important and in all the examples presented in this paper the initial guess for all of the variables was set to be either 0 or 1. Further, Example 3 is a very severe test of the robustness of the minimal energy technique and the results obtained for this problem are most encouraging.

Finally, it should be noted that the matrix $W$ in condition (3.19) is a full matrix and, therefore, a large amount of computer storage is required when the final time, $T_0$, is large or in the application of the method to higher dimensions. In order to reduce this large storage requirement then an alternative method which uses a time-marching scheme may be appropriate. Such a scheme is at present being investigated.

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